Math 1020 Week 7

Exponential Function

Defn Let $C \neq 0$, $a > 0$, $a \neq 1$

$$f(x) = Ca^x$$

is called an exponential function.

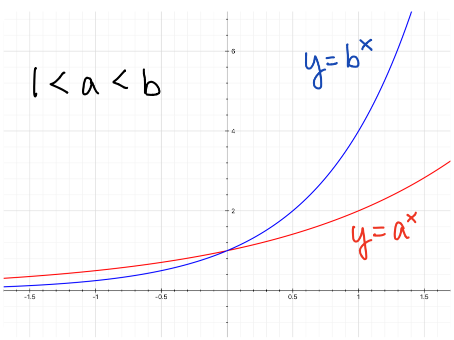
Note $f(0) = C$

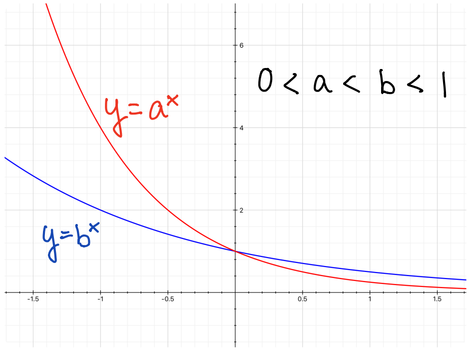
$$f(x+1) = Ca^{x+1}$$
$$= Ca^x \cdot a$$
$$= a f(x)$$

C = initial value

a = growth factor

Graph of Exponential functions





For $a > 1$, $\lim_{x\to-\infty} a^x = 0$

$\lim_{x\to\infty} a^x = \infty$

a^x is increasing

For $0 < a < 1$ $\lim_{x\to-\infty} a^x = \infty$

$\lim_{x\to\infty} a^x = 0$

a^x is decreasing

Properties of $f(x) = a^x$

- Domain $= \mathbb{R}$, Range $= (0, \infty)$
- No x-intercept, y-intercept = 1
- has horizontal asymptote $y = 0$ but no vertical asymptote
- One-to-one
- It passes through points $(0,1)$, $(1, a)$, $(-1, \frac{1}{a})$

Logarithm

Let $a>0$, $a\neq 1$

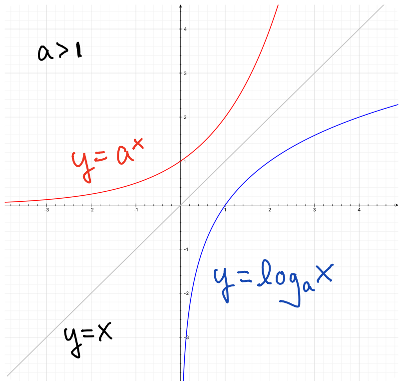
Then a^x is one-to-one

$\therefore a^x$ has inverse

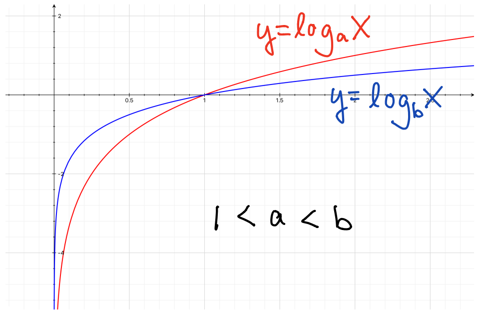
$\boxed{\begin{array}{l}\underline{\text{Defn}}\ \text{Define} \\ \qquad f(x) = \log_a x \\ \text{to be the inverse of } a^x\end{array}}$

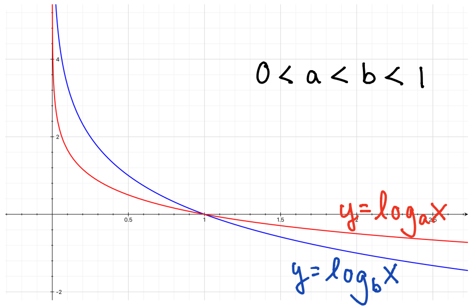
$$a^x = y \longleftrightarrow \underbrace{\log_a y}_{} = x$$

"What power of a is y"



Graph of Logarithmic functions





For $a>1$, $\lim_{x\to 0^+} \log_a x = -\infty$

$\lim_{x\to\infty} \log_a x = \infty$

$\log_a x$ is increasing

For $0<a<1$ $\lim_{x\to 0^+} \log_a x = \infty$

$\lim_{x\to\infty} \log_a x = -\infty$

$\log_a x$ is decreasing

Properties of $f(x) = \log_a x$

- Domain $=(0,\infty)$, Range $=\mathbb{R}$
- x-intercept $=1$, No y-intercept
- has vertical asymptote $x=0$ but no horizontal asymptote
- One-to-one
- It passes through points $(1,0)$, $(a,1)$, $(\frac{1}{a},-1)$

Recall: $\boxed{\underline{\text{Defn}}\quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
10	2.5937....
1000	2.7169...
100000	2.718267..

$e \approx 2.718281828459045\ldots$

Some other fact about e

- $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$
- $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

Important bases

- $\log x = \log_{10} x$ (common log)
- $\ln x = \log_e x$ (natural log)

Formula Let $a>0$, $a\neq 1$. Then

Exponential		Logarithmic
$a^{\log_a x} = x$	$\Longleftrightarrow$	$\log_a a^x = x$
$a^1 = a$		$\log_a a = 1$
$a^0 = 1$		$\log_a 1 = 0$
$a^x \cdot a^y = a^{x+y}$		$\log_a(xy) = \log_a x + \log_a y$
$\frac{a^x}{a^y} = a^{x-y}$		$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$		$\log_a(x^y) = y\log_a x$
		$\log_x y = \frac{\log_a y}{\log_a x}$

Ex Prove log formula from exp formula

eg

$$a^{y\log_a x} = \left(a^{\log_a x}\right)^y = x^y \Rightarrow \log_a x^y = y\log_a x$$

Exercises

① Express $\log 72$ and $\log 75$ in terms of $a = \log 2$ and $b = \log 3$

Sol

Note $72 = 2^3 \cdot 3^2$

$$\Rightarrow \log 72 = \log(2^3 \cdot 3^2) = 3\log 2 + 2\log 3 = 3a + 2b$$

Also, $75 = 3 \cdot 5^2 = 3 \cdot \left(\frac{10}{2}\right)^2 = \frac{3 \cdot 10^2}{2^2}$

$$\Rightarrow \log 75 = \log 3 + 2\log 10 - 2\log 2 = b - 2a + 2$$

② Find implied (natural) domain of

$$f(x) = \ln\frac{1}{x-7} + 5\ln x - \frac{1}{3}\ln(100-x)$$

and express $f(x)$ in the form $\ln(g(x))$

Sol $\ln y$ is defined $\Leftrightarrow$ $y > 0$

$\therefore$ Need $\frac{1}{x-7}$, x, $100 - x > 0$

$\Rightarrow$ $x > 7$ and $x > 0$ and $x < 100$

$\therefore$ $D_f = (7, 100)$

For $7 < x < 100$,

$$f(x) = \ln\frac{1}{x-7} + \ln x^5 - \ln(100-x)^{\frac{1}{3}} = \ln\frac{x^5}{(x-7)(100-x)^{\frac{1}{3}}}$$

Rmk

$f(x)$ and $\underbrace{\ln\frac{x^5}{(x-7)(100-x)^{\frac{1}{3}}}}_{\text{Domain} = (-\infty, 0) \cup (7, 100) \neq D_f}$ are not exactly the same

③ Solve $3^{2x} = 7 \cdot 3^x + 18$

Sol Note $3^{2x} = (3^x)^2$

$$\therefore \quad (3^x)^2 = 7 \cdot 3^x + 18$$

$$(3^x)^2 - 7 \cdot 3^x - 18 = 0$$

$$(3^x - 9)(3^x + 2) = 0$$

$$\Rightarrow \quad 3^x = 9 \quad \text{or} \quad 3^x = -2 \quad \left(\begin{array}{l}\text{no solution} \\ \because\ 3^x > 0\end{array}\right)$$

$$3^x = 3^2$$

3^x is one-to-one $\Rightarrow x = 2$

Compare: $y^2 = 4 = 2^2$

$\cancel{\Rightarrow}\ y = 2$ (y may be -2)

Reason: y^2 is not one-to-one

④ Solve $7^{x+2} = 9^{2x-5}$

Sol

$$\log 7^{x+2} = \log 9^{2x-5}$$

$$(x+2)\log 7 = (2x-5)\log 9$$

$$x(\log 7 - 2\log 9) = -2\log 7 - 5\log 9$$

$$\Rightarrow \quad x = \frac{-2\log 7 - 5\log 9}{\log 7 - 2\log 9}$$

Rmk

$$x = \frac{-2\ln 7 - 5\ln 9}{\ln 7 - 2\ln 9}$$ is also an answer

⑤ Let $f(x) = \log(5x-1)$. Find $f^{-1}(x)$.

Determine domain and range of f and f^{-1}

<u>Sol</u> Let $y = f(x) = \log(5x-1)$

Then $10^y = 5x-1$

$10^y + 1 = 5x$

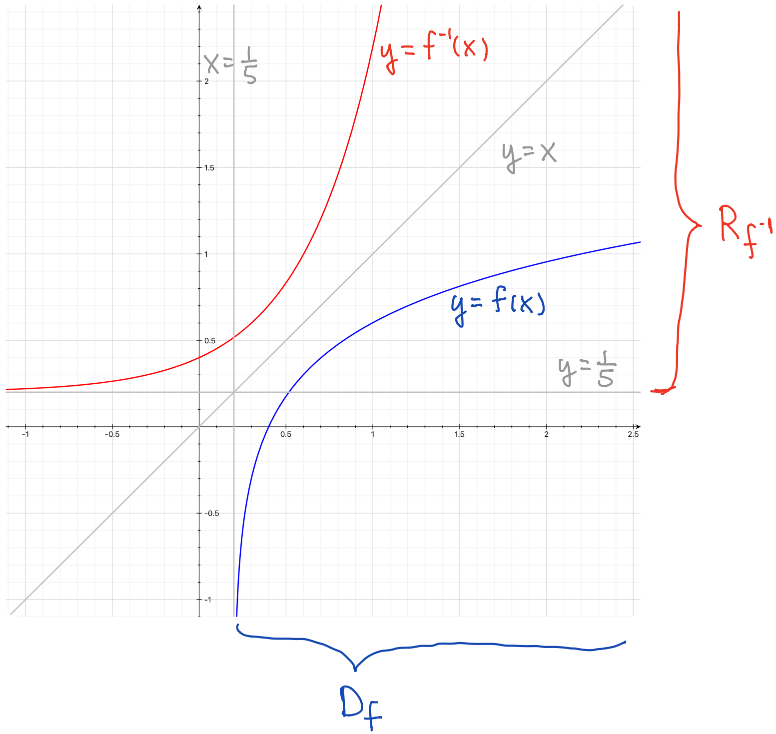
$$x = \frac{10^y+1}{5} = f^{-1}(y)$$

$$\therefore f^{-1}(x) = \frac{10^x+1}{5}$$

For D_f, need $5x-1>0 \Rightarrow x > \frac{1}{5}$

$$\therefore R_{f^{-1}} = D_f = \left(\tfrac{1}{5}, \infty\right)$$

Also, $R_f = D_{f^{-1}} = \mathbb{R} = (-\infty, \infty)$



$y=f(x)$ has vert. asymptote $x=\frac{1}{5}$ $\left(\lim_{x\to\frac{1}{5}} f(x) = -\infty\right)$

$\Updownarrow$

$y=f^{-1}(x)$ has hor. asymptote $y=\frac{1}{5}$ $\left(\lim_{x\to-\infty} f^{-1}(x) = \frac{1}{5}\right)$

⑥ Let $f(x) = \dfrac{10e^x}{1+e^x}$.

Find $f^{-1}(x)$, range of f and f^{-1}.

Sol Let $y = f(x) = \dfrac{10e^x}{1+e^x}$

then
$$y(1+e^x) = 10e^x$$
$$e^x(y-10) = -y$$
$$e^x = \frac{y}{10-y}$$
$$x = \ln\left(\frac{y}{10-y}\right) = f^{-1}(y)$$

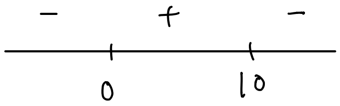
$\therefore\ f^{-1}(x) = \ln\left(\dfrac{x}{10-x}\right)$

$f(x)$ is defined for any $x \in \mathbb{R}$

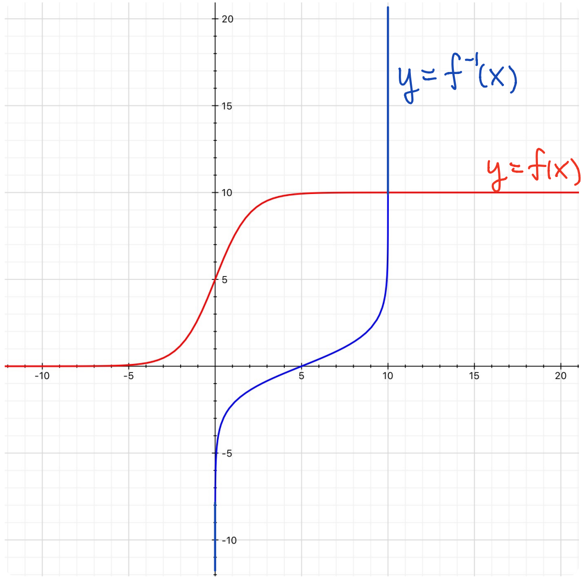
$\therefore\ R_{f^{-1}} = D_f = \mathbb{R}$

For $D_{f^{-1}}$, need $\dfrac{x}{10-x} > 0$

$\Rightarrow\ 0 < x < 10$



$\therefore\ R_f = D_{f^{-1}} = (0, 10)$



$y = f(x)$ has horizontal asymptotes $y = 0,\ y = 10$

$y = f^{-1}(x)$ has vertical asymptotes $x = 0,\ x = 10$

Graph by transformation

a. $y = 2^{-x} - 1$

2^x

$\downarrow$ Reflect across y-axis

2^{-x}

$\downarrow$ $\downarrow$ 1 unit

$2^{-x} - 1$

b. $y = 3\left(\frac{1}{2}\right)^{2x}$

$\left(\frac{1}{2}\right)^x$

$\downarrow$ horizontal scaling $\times \frac{1}{2}$

$\left(\frac{1}{2}\right)^{2x}$

$\downarrow$ vertical scaling $\times 3$

$3\left(\frac{1}{2}\right)^{2x}$

c. $y = -\ln(x+3)$

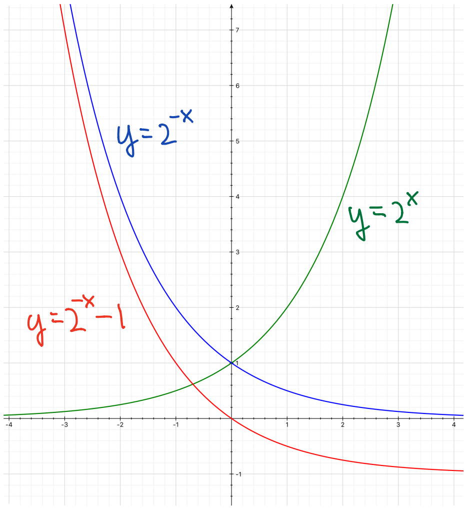
$\ln x$

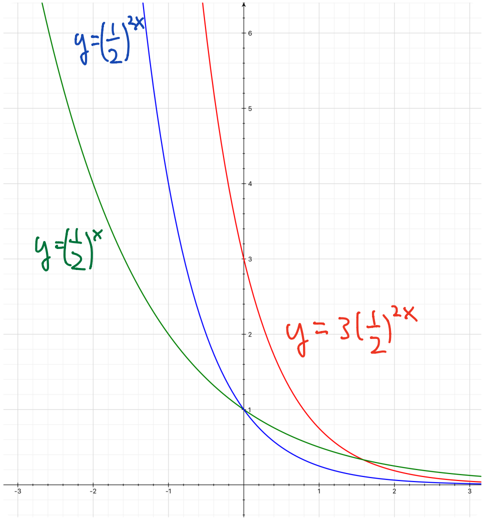
$\downarrow$ $\leftarrow$ 3 units

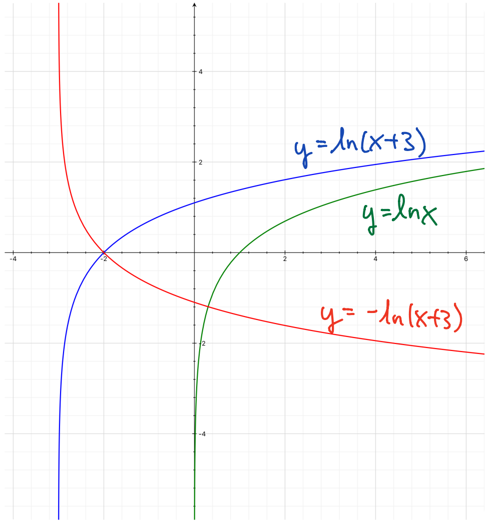
$\ln(x+3)$

$\downarrow$ Reflect across x-axis

$-\ln(x+3)$







Partial fractions

Goal: Express a proper rational function as a sum of simpler ones

Rmk: $\frac{p(x)}{q(x)}$ is proper if $\deg p < \deg q$

eg

$$\frac{x}{x^2+3x+2} = \frac{-1}{x+1} + \frac{2}{x+2}$$

$$\frac{x^2+20x+11}{(x+1)^2(x-3)} = \frac{-4}{x+1} + \frac{2}{(x+1)^2} + \frac{5}{x-3}$$

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \underbrace{\frac{-x+7}{x^2+x+1} + \frac{5}{x-2}}$$

Rmk RHS is easier for integration

Recall

ax^2+bx+c is irreducible (cannot be further factorized)

$\iff \Delta = b^2-4ac<0$

- $x^2-1=(x+1)(x-1)$ is reducible $(\Delta=4)$
- x^2+x+10 is irreducible $(\Delta=-39)$

Procedure: Given proper $\frac{p(x)}{q(x)}$

① Factorize $q(x)$ into a product of linear and irreducible quadratic factors

② Write down general terms

Factor of $q(x)$	Terms in partial fractions
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
irreducible	
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

③ Determine the unknown coefficients A_i, B_i in step ② by substitution or comparing coefficients.

eg $\dfrac{9x-13}{x^2+x-12}$

① $x^2+x-12=(x+4)(x-3)$

② General terms:

Let $\dfrac{9x-13}{x^2+x-12}=\dfrac{A}{x+4}+\dfrac{B}{x-3}$

③ $\Rightarrow 9x-13 = A(x-3)+B(x+4)$

$= (A+B)x+(-3A+4B)$

Comparing coefficients

$$\Rightarrow \begin{cases} A+B=9 & \cdots ① \\ -3A+4B=-13 & \cdots ② \end{cases}$$

$3\times① + ② \Rightarrow 7B=14 \Rightarrow B=2$

Put $B=2$ into ① $\Rightarrow A=7$

$$\therefore \frac{9x-13}{x^2+x-12}=\frac{7}{x+4}+\frac{2}{x-3}$$

eg $\dfrac{x^2+20x+11}{(x+1)^2(x-3)}$ ← already factorized

Sol General term:

Let $\dfrac{x^2+20x+11}{(x+1)^2(x-3)}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x-3}$

$$x^2+20x+11=A(x+1)(x-3)+B(x-3)+C(x+1)^2$$

We determine coefficients by substitution this time:

Put $x=3$ $\quad 80=16C \Rightarrow C=5$

Put $x=-1$ $\quad -8=-4B \Rightarrow B=2$

Put $x=0$ $\quad 11=-3A-3B+C=-3A-1$

$\Rightarrow A=-4$

$$\frac{x^2+20x+11}{(x+1)^2(x-3)}=-\frac{4}{x+1}+\frac{2}{(x+1)^2}+\frac{5}{x-3}$$

eg $\dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)}$

$\underbrace{x^2+x+1}$ $\Delta = -3 \Rightarrow$ irreducible

Sol General terms.

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

$$4x^2+14x-9 = (Ax+B)(x-2) + C(x^2+x+1)$$

Substitutions (eg $x=2,0,1$) give

3 equations $\xrightarrow{\text{Solve}}$ $A=-1, B=7, C=5$

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \frac{-x+7}{x^2+x+1} + \frac{5}{x-2}$$

eg $\dfrac{x^4+3x^2-x+1}{x^5+2x^3+x}$

Sol $x^5+2x^3+x = x(x^4+2x^2+1) = x(x^2+1)^2$

Let $\dfrac{x^4+3x^2-x+1}{x^5+2x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

By substitution/comparing coefficients + Solving eqn :

$$\frac{x^4+3x^2-x+1}{x^5+2x^3+x} = \frac{1}{x} + \frac{x-1}{(x^2+1)^2} \quad (B=C=0)$$

Rmk If $\dfrac{p(x)}{g(x)}$ is improper $(\deg p \geq \deg g)$

we should do long division first

eg $\dfrac{2x^3}{x^2-1} = 2x + \dfrac{2x}{x^2-1} = 2x + \dfrac{1}{x-1} + \dfrac{1}{x+1}$

Long division

Partial fractions

$\Rightarrow 2x^3 = (x^2-1)(2x) + 2x$